Stochastic Geometry
&
Wireless Networks Densification

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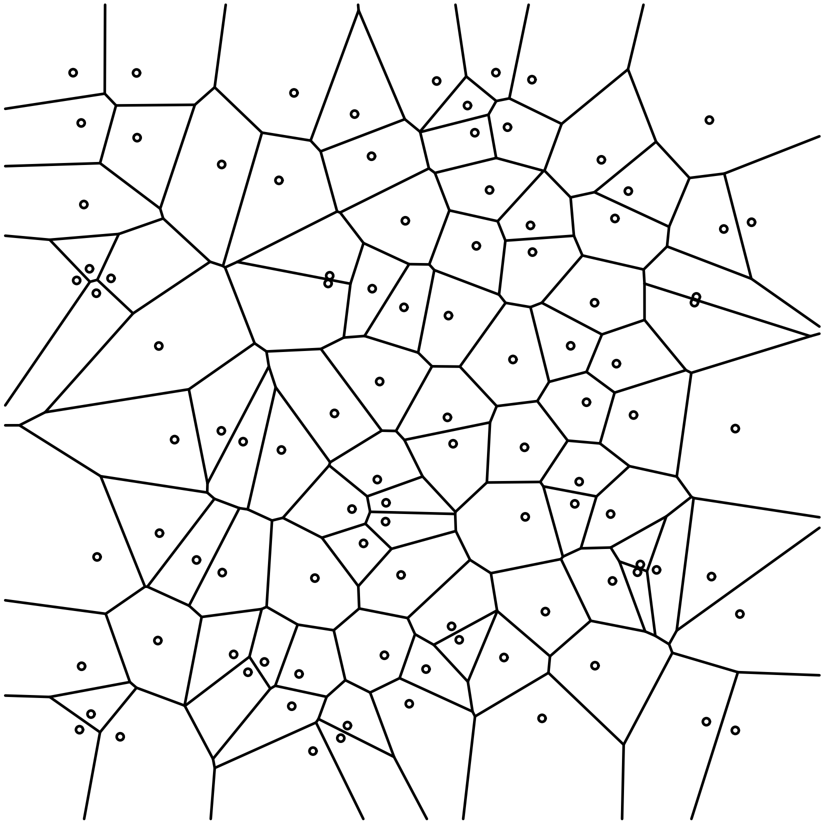
STRUCTURE OF THE TALK

- **Dense Poisson Cellular Networks and their Backhaul**
- **Dense Poisson Device2Device Networks**
- **Dense Poisson Multiuser Information Theory Networks**

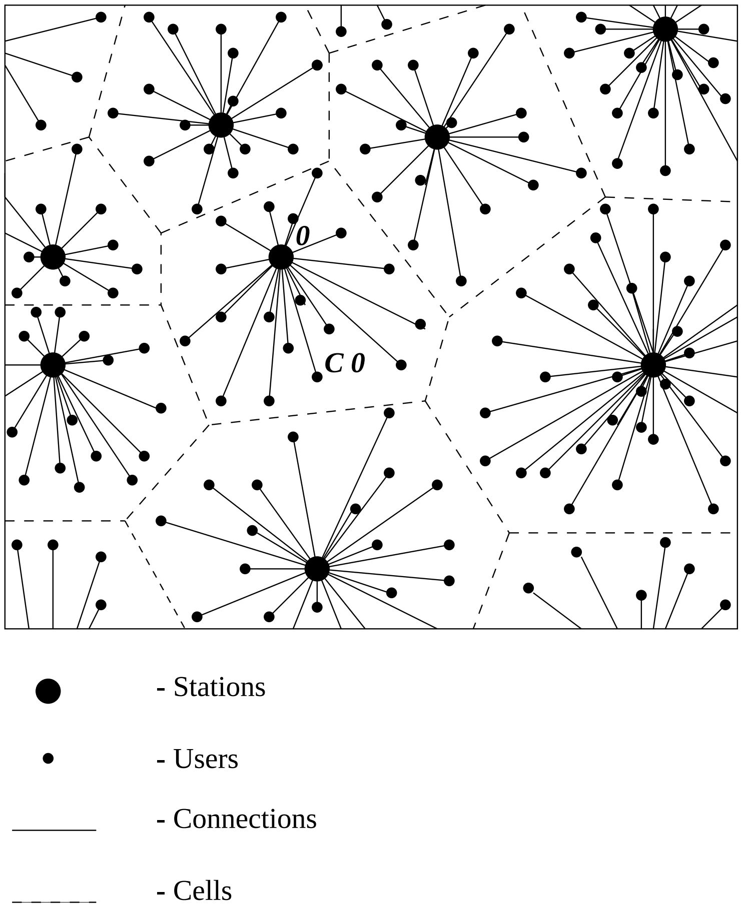
VORONOI TESSELLATION



Voronoi tessellation generated by a random sample of points.

- Φ set of points of the Euclidean plane.
- The **Voronoi cell** $\mathcal{C}_{x_i}(\Phi)$ of atom x_i of Φ is the set of all locations of $\mathbb{R}^2$ that are closer to this atom x_i than to any other atom of Φ.
- Each Voronoi cell is a **convex polyhedron**, but it may be unbounded.

POISSON-VORONOI CELLULAR NETWORKS



- **Base stations** (BSs) arranged according to an homogeneous Poisson point process of intensity λ in $\mathbb{R}^2$
- **Users**
  - located according to some independent stationary point process
  - each user is served with the closest BS $\rightarrow$ **Poisson Voronoi Cells**

SHANNON RATE IN POISSON-VORONOI NETWORKS
IEEE RICE

- **Focus**: **Downlink**
- **SINR experienced by tagged user**:
  - **Signal**: **stems from closest BS**
  - **Interference**: **stems from BSs outside Voronoi cell of tagged user**
  - **Thermal noise**: **with power** N
- **Shannon rate offered to tagged user**:

$$\mathbf{B}\log(\mathbf{1}+\mathbf{SINR})$$

- **Question**: **Law of the Shannon rate offered to tagged user**

PROPAGATION ASSUMPTIONS

- **Power law path loss model: at distance r**

$$\mathbf{l(r) = r^{\beta}}$$

with $\beta > 2$ the path loss exponent

- **Other classical path loss model can be treated as well.**

FADING ASSUMPTIONS

- **Simplest setting:**
  - **Fading on the downlink from BS to tagged user:**
    Rayleigh with representative S with mean $\frac{1}{\mu}$ with $\frac{1}{\mu} = \mathrm{P}_{\mathrm{tx}}$
  - **Fading from other BSs:**
    i.i.d. with representative F with general distribution.
- **More general assumptions can be handled as well**

COVERAGE/SHANNON RATE/SPECTRAL EFFICIENCY

- **Coverage probability seen from a typical user**

$$\mathbf{p_c}(\mathbf{T}, \lambda, \beta) = \mathbf{P^0_u}[\mathbf{SINR} > \mathbf{T}] = \mathbf{P^0_u}[\text{Shannon rate} > \mathbf{B}\log(\mathbf{1} + \mathbf{T})]$$

- **Equivalent to**

– Probability that a randomly chosen user achieves target SINR T

– Average fraction of users who at any time achieve SINR T

– Average fraction of the network area in "T-coverage" at any time

MAIN RESULT

- **Theorem**

$$\mathbf{p_c}(\mathbf{T},\lambda,\beta) = \pi\lambda \int_{\mathbf{0}}^{\infty} \mathbf{e}^{-\pi\lambda\mathbf{v}\kappa(\mathbf{T},\beta)-\mu\mathbf{T}\mathbf{N}\mathbf{v}^{\beta/\mathbf{2}}}\mathrm{d}\mathbf{v}$$

where

$$\kappa(\mathbf{T},\beta) = \frac{\mathbf{2}(\mu\mathbf{T})^{\frac{\mathbf{2}}{\beta}}}{\beta}\mathbb{E}\left[\mathbf{F}^{\frac{\mathbf{2}}{\beta}}\left(\boldsymbol{\Gamma}(-\mathbf{2}/\beta,\mu\mathbf{T}\mathbf{F}) - \boldsymbol{\Gamma}(-\mathbf{2}/\beta)\right)\right]$$

and the expectation is with respect to the fading F and

$$\boldsymbol{\Gamma}(\mathbf{w},\mathbf{z}) = \int_{\mathbf{t}=\mathbf{z}}^{\infty} \exp(-\mathbf{t})\mathbf{t}^{\mathbf{w}-\mathbf{1}}\mathbf{dt}, \quad \boldsymbol{\Gamma}(\mathbf{w}) = \int_{\mathbf{t}=\mathbf{0}}^{\infty} \exp(-\mathbf{t})\mathbf{t}^{\mathbf{w}-\mathbf{1}}\mathbf{dt} = \boldsymbol{\Gamma}(\mathbf{w},\mathbf{0}).$$

$\boxed{\text{PROOF}}$

- **Step 1: Poisson-Voronoi**

$$\mathbf{P}(\Phi(B(0,r)) = 0) = e^{-\lambda\pi r^2}$$

$$\begin{aligned}
\mathbf{p_c}(\mathbf{T},\lambda,\beta) &= \mathbf{P}[\mathbf{SINR} > \mathbf{T}] \\
&= \int_{\mathbf{r}>\mathbf{0}} \mathbf{P}[\mathbf{SINR} > \mathbf{T}]\mathbf{f_r}(\mathbf{r})\mathrm{d}\mathbf{r} \\
&= \int_{\mathbf{r}>\mathbf{0}} \mathbf{P}\left[\frac{\mathbf{S}\mathbf{r}^{-\beta}}{\mathbf{N}+\mathbf{I_r}} > \mathbf{T}\right]\mathbf{e}^{-\pi\lambda\mathbf{r}^2}\mathbf{2}\pi\lambda\mathbf{r}\mathrm{d}\mathbf{r} \\
&= \int_{\mathbf{r}>\mathbf{0}} \mathbf{e}^{-\pi\lambda\mathbf{r}^2}\mathbf{P}[\mathbf{S} > \mathbf{T}\mathbf{r}^{\beta}(\mathbf{N}+\mathbf{I_r})]\mathbf{2}\pi\lambda\mathbf{r}\mathrm{d}\mathbf{r}
\end{aligned}$$

$\mathbf{I_r}$: interference power given the closest BS is at distance r

PROOF (*continued*)

- **Step 2: Rayleigh. Since** $\mathbf{S} \sim \exp(\mu)$**,**

$$\mathbf{P}(\mathbf{S} > \mathbf{T}\mathbf{r}^{\beta}(\mathbf{N} + \mathbf{I}_{\mathbf{r}})) = \mathbb{E}[\exp(-\mu \mathbf{T}\mathbf{r}^{\beta}(\mathbf{N} + \mathbf{I}_{\mathbf{r}})] = \mathbf{e}^{-\mu \mathbf{T}\mathbf{r}^{\beta}\mathbf{N}} \mathcal{L}_{\mathbf{I}_{\mathbf{r}}}(\mu \mathbf{T}\mathbf{r}^{\beta})$$

with $\mathcal{L}_{\mathbf{I}_{\mathbf{r}}}(\mathbf{s})$ **the Laplace transform of the interference.**
Thus

$$\mathbf{p_c}(\mathbf{T}, \lambda, \beta) = \int\limits_{\mathbf{r}>\mathbf{0}} \mathbf{e}^{-\pi\lambda\mathbf{r}^{\mathbf{2}}} \mathbf{e}^{-\mu \mathbf{T}\mathbf{r}^{\beta}\mathbf{N}} \mathcal{L}_{\mathbf{I}_{\mathbf{r}}}(\mu \mathbf{T}\mathbf{r}^{\beta}) \mathbf{2}\pi\lambda\mathbf{r}\mathrm{d}\mathbf{r}$$

PROOF (*continued*)

- **Step 3: Interference as Poisson Shot Noise Field**

$$\mathcal{L}_{\mathbf{I_r}}(\mathbf{s}) = \exp\left(-\mathbf{2}\pi\lambda \int_{\mathbf{r}}^{\infty} \left(\mathbf{1} - \mathcal{L}_{\mathbf{F}}\left(\mathbf{s}\mathbf{v}^{-\beta}\right)\right) \mathbf{v}\mathrm{d}\mathbf{v}\right)$$

with $\mathcal{L}_F$ the Laplace Transform of the general fading.

$$\mathcal{L}_{\mathbf{I_r}}(\mu\mathbf{T}\mathbf{r}^{\beta}) \;=\; \exp\left(\lambda\pi\mathbf{r}^{\mathbf{2}} - \frac{\mathbf{2}\pi\lambda(\mu\mathbf{T})^{\frac{\mathbf{2}}{\beta}}\mathbf{r}^{\mathbf{2}}}{\beta}\mathbf{Z}\right)$$

with

$$\mathbf{Z} = \int_{\mathbf{0}}^{\infty} \mathbf{g}^{\frac{\mathbf{2}}{\beta}} \left[\boldsymbol{\Gamma}(-\mathbf{2}/\beta, \mu\mathbf{T}\mathbf{g}) - \boldsymbol{\Gamma}(-\mathbf{2}/\beta)\right] \mathbf{f}(\mathbf{g})\mathrm{d}\mathbf{g}$$

with $\mathrm{f}(\mathrm{g})$ the PDF of F.

SPECIAL CASES

- **Special Case: Rayleigh F, $\beta = 4$:**

$$\mathbf{p_c}(\mathbf{T}, \lambda, \mathbf{4}) = \frac{\pi^{\frac{3}{2}}\lambda}{\sqrt{\mathbf{T}/\mathbf{SNR}}} \exp\left(\frac{(\lambda\pi\nu(\mathbf{T}))^{\mathbf{2}}}{\mathbf{4T}/\mathbf{SNR}}\right) \mathbf{Q}\left(\frac{\lambda\pi\nu(\mathbf{T})}{\sqrt{\mathbf{2T}/\mathbf{SNR}}}\right)$$

where

$$\begin{aligned}\nu(\mathbf{T}) &= \mathbf{1} + \sqrt{\mathbf{T}}(\pi/\mathbf{2} - \arctan(\mathbf{1}/\sqrt{\mathbf{T}})) \\ \mathbf{SNR} &= \frac{\mathbf{1}}{\mu \mathrm{N}}\end{aligned}$$

SPECIAL CASES (*continued*)

- **Interference Limited, Rayleigh F, $\beta = 4$:**

$$\mathbf{p_c(T, \lambda, 4)} = \frac{\mathbf{1}}{\mathbf{1} + \sqrt{\mathbf{T}}(\pi/\mathbf{2} - \arctan(\mathbf{1}/\sqrt{\mathbf{T}}))}$$

- **If the user and BS densities scale the same way,**

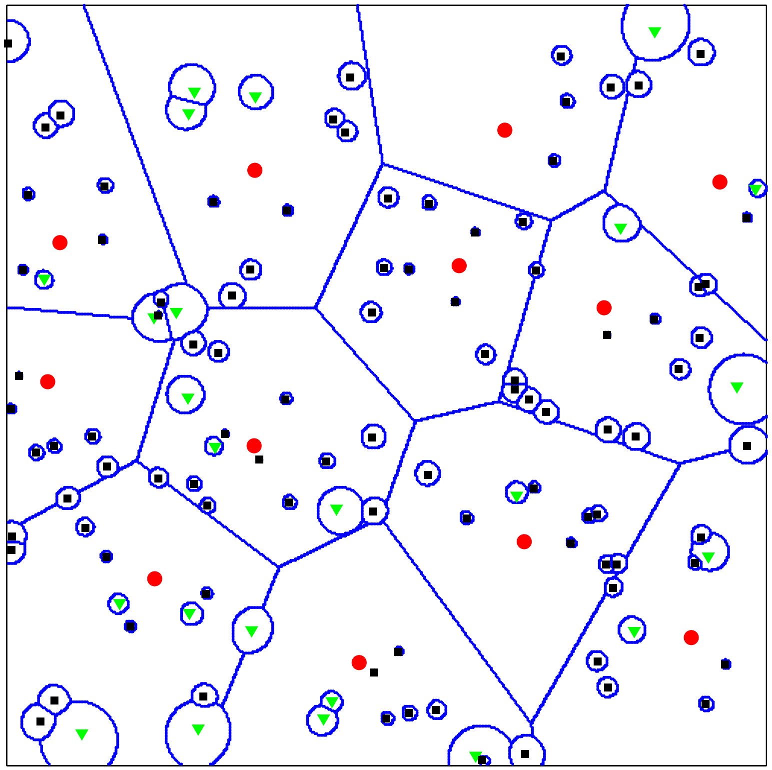
- **positive spectral efficiency for all densities !**

MULTI-TIER VARIANT– IEEE ABRAHAM

- **Network Elements**
  1. **Macro BS**
  2. **Pico BS**
  3. **Femto BS**
- **BS of type i with power P_i with**

$$P_1 > P_2 > P_3$$

- **Closed form expression for Probability of coverage in CBR and Shannon rate distribution**



COVERAGE

- **Tier i:**
  - **Poisson point process Φ_i of intensity λ_i in $\mathbb{R}^2$**
  - **Transmit power P_i, SINR Target T_i**
- **Power law path loss, Rayleigh fading**
- **Coverage of the tagged customer:**

$$\max_{\mathbf{i}} \max_{\mathbf{x} \in \Phi_{\mathbf{i}}} \mathbf{SINR}(\mathbf{x} \to \mathbf{0}) > \mathbf{T_i}$$

SPECTRAL EFFICIENCY

- **Theorem**

Assume $T_i > 1$ for all i. Then

$$\mathbf{p_c} = \sum_{\mathbf{i}} \lambda_{\mathbf{i}} \int_{\mathbb{R}^2} \mathbf{e}^{-\left(\frac{\mathbf{T_i}}{\mathbf{P_i}}\right)^{\frac{2}{\beta}} \mathbf{C}(\beta)||\mathbf{x_i}||^2 \left(\sum_{\mathbf{m}} \lambda_{\mathbf{m}} (\mathbf{P_m})^{\frac{2}{\beta}}\right)} \mathbf{e}^{-\frac{\mathbf{T_i N}}{\mathbf{P_i}}||\mathbf{x_i}||^{\beta}} \mathbf{dx_i}$$

with

$$\mathbf{C}(\beta) = \frac{\mathbf{2}\pi^{\mathbf{2}}}{\sin(\frac{\mathbf{2}\pi}{\beta})\beta}$$

SPECTRAL EFFICIENCY (*continued*)

- **Interference limited special case**

$$\mathbf{p_c} = \frac{\pi}{\mathbf{C}(\beta)} \frac{\sum_{\mathbf{i}} \lambda_{\mathbf{i}} \left(\frac{\mathbf{P_i}}{\mathbf{T_i}}\right)^{\frac{2}{\beta}}}{\sum_{\mathbf{i}} \lambda_{\mathbf{i}} (\mathbf{P_i})^{\frac{2}{\beta}}}$$

- **If the user and all BS densities scale the same way,**

- **positive spectral efficiency for all densities !**

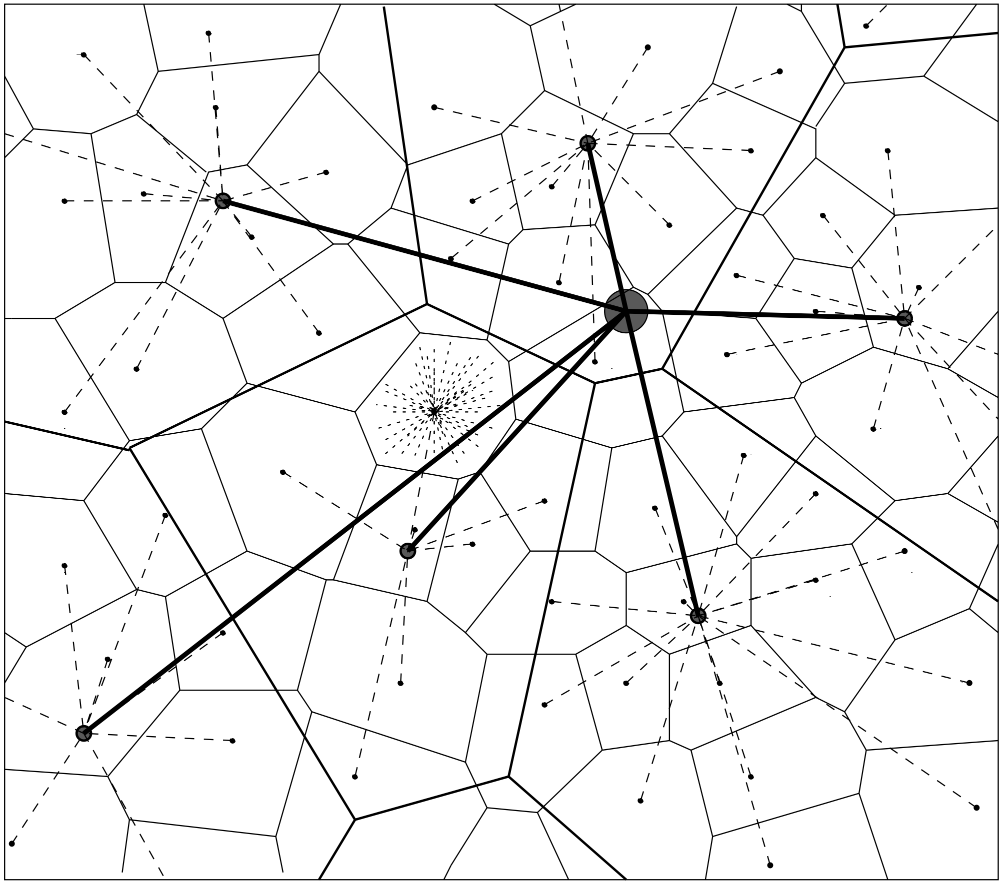
BACKHAUL INFRASTRUCTURE COST

- **No free lunch: densification leads to explosion of infrastructure cost**
  - **What is the backhaul architecture that minimizes connection costs per unit of space?**
  - **What is the optimum number of levels?**
  - **What should be the relative intensities of the BS's of each level?**
- **Example of SG Analysis: Hierarchical architectures**

HIERARCHICAL POISSON–VORONOI MODEL

- $\Phi_i,\ i = 1, \ldots, N$ **point processes describing positions of BS's of type** i
- **Poisson Model: Φ_i are independent homogeneous Poisson point processes of intensity λ_i ($\lambda_i > \lambda_{i+1}$).**
- **Hierarchical connections the i-th level stations in a cell of a $(i+1)$-st level station are directly connected to the latter.**

HIERARCHICAL POISSON–VORONOI MODEL (*continued*)



ANALYTICAL RESULTS

- **Additive functionals**

$$\mathbb{E}^0_{i+1}\left(\sum_{x_j\in\Phi_i} f(x_j)\mathbf{1}\{x_j\in\mathcal{C}^0_{i+1}\}\right)$$

for $f:\mathbb{R}^2\to\mathbb{R}_+$**, where** $\mathcal{C}^0_{i+1}$ **is the cell of** $0\in\Phi_{i+1}$

- **Examples:**
  - $\mathcal{N}_i$ **number of type** i **BS's in** $\mathcal{C}^0_{i+1}$ **(**$f(x)=1$**);**
  - $\mathcal{L}_i(a)$ **cost of all connections in** $\mathcal{C}^0_{i+1}$**, if the cost for connecting a type** i **BS at** x **to its type** $i+1$ **BS at 0 is**

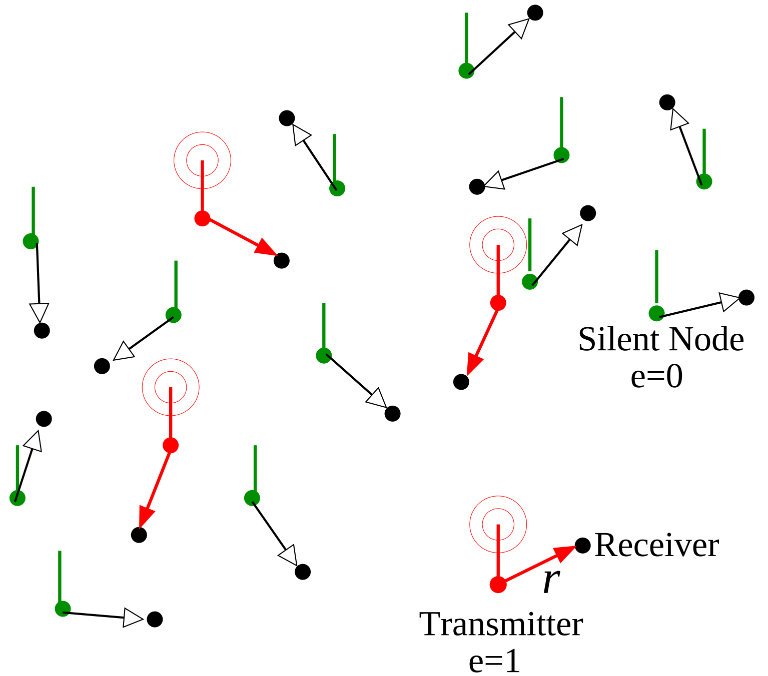
$$f(x)=|x|^a,\ a\geq 1$$

- **Explicit integral expressions for many additive functionals.**

DEVICE TO DEVICE NETWORKS

- **Aloha, SG analysis, dense Aloha case**
- **CSMA, SG analysis, dense CSMA case QUALCOM FLASHLINK**

ALOHA ON A POISSON DIPOLE D2D NETWORK



Dipoles (transmitter,receiver)
Here, fixed length r, uniform direction dipoles.

- **Nodes form a Poisson p.p. in the Euclidean plane**
- **Each node uses Aloha**
- **Transm. success depends on SINR at the receiver**

INTERFERENCE AS POISSON SHOT NOISE

- $\Phi = \{X_i\}$: **transmitters locations: Poisson with intensity** λ
- $e_i \in \{0, 1\}$: **right for** i **to access medium: i.i.d. with** $P(e_i = 1) = p$
- $F_{i,y} \in \mathbb{R}_+$: **fading from transmitter** i **to location** y**: Rayleigh** μ
- $I_{\Phi^e}(y) = \sum_{i \neq 0} \frac{e_i F_{i,y}}{|y - X_i|^\beta}$: **'filtered' interference at y,**
- **Node at** 0 **active in slot can be received by that located at y iff**

$$\mathrm{SINR}(\mathbf{y}) = \frac{\mathbf{F_{0,y}}/|\mathbf{y}|^\beta}{\mathbf{N} + \mathbf{I}_{\phi^e}(\mathbf{y})} \geq \mathbf{T} \quad [\mathrm{Outage}]$$

COVERAGE/SPECTRAL EFFICIENCY

- **The probability of coverage at distance r is**

$$\mathbf{p_c(r)} \;=\; \mathbb{P}(\mathbf{F} \geq \mathbf{Tr}^{\beta}(\mathbf{N} + \mathbf{I}_{\Phi^e})) = \mathcal{L}_{\mathbf{N}}\left(\mu\mathbf{Tr}^{\beta}\right)\mathcal{L}_{\mathbf{I^e}}\left(\mu\mathbf{Tr}^{\beta}\right)$$
$$\;=\; \mathcal{L}_{\mathbf{N}}\left(\mu\mathbf{Tr}^{\beta}\right)\exp\left\{-\mathbf{2}\pi\lambda\mathbf{p}\int_{\mathbf{0}}^{\infty}\frac{\mathbf{u}}{\mathbf{1}+\mathbf{u}^{\beta}/(\mathbf{Tr}^{\beta})}\mathbf{du}\right\}$$

with $\mathcal{L}_{\mathrm{A}}$ the Laplace transform of the positive random variable A.

- **Example: If N $\equiv$ 0, then**

$$\mathbf{p_c(r)} = \exp(-\lambda\mathbf{pr^2T}^{\mathbf{2}/\beta}\mathbf{C}(\beta)), \qquad \mathbf{C}(\beta) = \frac{\mathbf{2}\pi^{\mathbf{2}}}{\beta\sin(\mathbf{2}\pi/\beta)}$$

POISSON D2D NETWORK DENSIFICATION

- **Theorem α-stable white noise field limit**
  When λ tends to infinity, the joint law of the rescaled interference field

$$\left(\frac{\mathbf{I}(\mathbf{x_1})}{(\lambda\mathbf{p})^{\frac{\beta}{\mathbf{2}}}}, \ldots, \frac{\mathbf{I}(\mathbf{x_k})}{(\lambda\mathbf{p})^{\frac{\beta}{\mathbf{2}}}}\right)$$

  converges in distribution to $(\xi_1, \ldots, \xi_k)$, an i.i.d. vector with

$$\mathbb{E}[\mathbf{e}^{-\mathbf{t}\xi_1}] = \exp\left(-\mathbf{C}(\beta)\mathbb{E}[\mathbf{F}^{\frac{\mathbf{2}}{\beta}}]\mathbf{t}^{\frac{\mathbf{2}}{\beta}}\right)$$

- **Corollary For a Poisson field of interferers with density λ, the scale at which the Shannon rate decreases for a link of length 1 is $\lambda^{-\frac{\beta}{\mathbf{2}}}$**

MULTI-USER INFORMATION THEORY

- **MIMO Mitigation of Interference under densification**
- **MAC Mitigation of Interference under densification**

MIMO MITIGATION

- **Poisson Dipole model - SIMO with local CSIR**
- **D2D transmitter as above**
- **D2D receiver**
  - **has A antennas**
  - **knows CSIR of direct link**
  - **uses MRC maximum ratio combining with CSIR**

SIMO SPECTRAL EFFICIENCY & SCALING LAW

- **Theorem** **In the interference limited regime, the mean Shannon rate of the typical user is**

$$\frac{\beta}{2\ln 2}\sum_{n=1}^{A}\binom{A}{n}\int_0^\infty \frac{e^{-u}}{u}\frac{\left(\frac{\mathrm{sinc}\left(\frac{2}{\beta}\right)u}{2\pi r^2}\right)^{n\beta/2}}{\left(1+\left(\frac{\mathrm{sinc}\left(\frac{2}{\beta}\right)u}{2\pi r^2}\right)^{\beta/2}\right)^{A}}du$$

- **Corollary** **Assume that** $A = c\lambda^\gamma$ **for some** $c > 0$ **and that there is no diversity limitation, then in the interference limited regime, when** $\lambda \to \infty$**, the spectral efficiency of the typical user scales like**

$$\log(1+\lambda^{\gamma-\frac{\beta}{2}})$$

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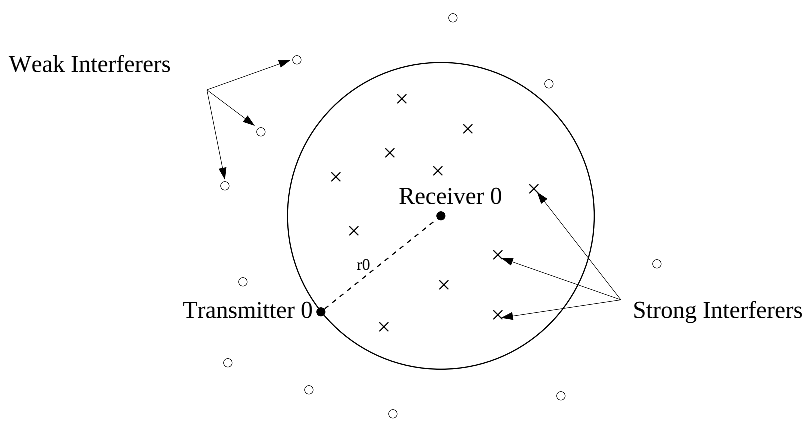
MAC SPECTRAL EFFICIENCY & SCALING LAW



The Multiple Access Channel "Multipole": the co–transmitters in the Multiple Access Channel are the strong interferers.

- **The capacity region of the Multiple Access Channel is known**
- **This capacity is achieved by simultaneous decoding**
- **The "channel order" k is a free parameter: SIM(k)**

RESULTS

- **Mean Shannon rate of typical MAC for given order**
- **Scaling law: for optimal MAC cardinality (order increasing linearly with λ), the mean Shannon rate is order 1 when $\lambda \to \infty$.**

CONCLUSION

- **SG provides a vertical integrator for**
  - **wireless networking**
  - **information theory, including multi-user**
  - **network architectures**
  - **densification**
  - **network economics**
- **SG is already used by manufacturers and operators**
- **SG has the potential to guide strategic decisions in 5G design**

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